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A Sequence of Triangles and Geometric Inequalities

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Abstract. We construct a sequence of triangles from a given one, and deduce a number of famous geometric inequalities.

1. A geometric construction

Throughout this paper we use standard notations of triangle geometry. Given a triangle ABC with sidelengths a, b, c, let s, R, r, and Δ denote the semiperimeter, circumradius, inradius, and area respectively. We begin with a simple geometric construction. Let H be the orthocenter of triangle ABC. Construct a circle, center H, radius $R' = \sqrt{2Rr}$ to intersect the half lines HA, HB, HC at A', B', C' respectively (see Figure 1).

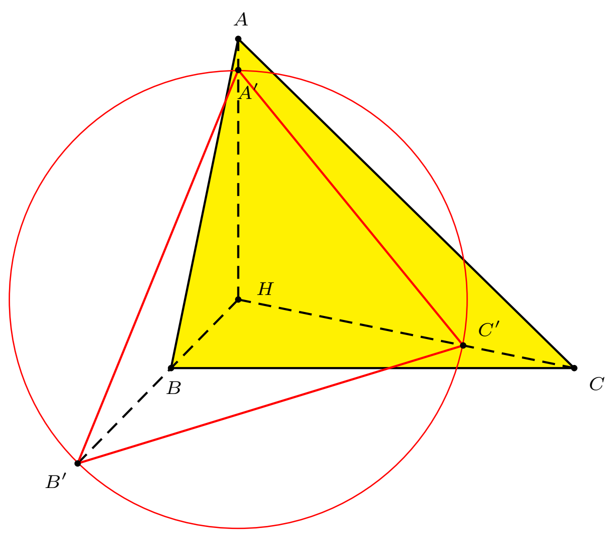

Figure 1.

If the triangle ABC has a right angle at A with altitude AD (D on the hypotenuse BC), we choose A' on the line AD such that A is between D and A'.

Lemma 1. *Triangle $A'B'C'$ has*
(a) *angle measures* $A' = \frac{\pi}{2} - \frac{A}{2}$, $B' = \frac{\pi}{2} - \frac{B}{2}$, $C' = \frac{\pi}{2} - \frac{C}{2}$,
(b) *sidelengths* $a' = \sqrt{a(b+c-a)}$, $b' = \sqrt{b(c+a-b)}$, $c' = \sqrt{c(a+b-c)}$, *and*
(c) *area* $\Delta' = \Delta$.

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Proof. (a) $\angle B'A'C' = \frac{1}{2}\angle B'HC' = \frac{1}{2}\angle BHC = \frac{\pi-A}{2}$; similarly for B' and C'.

(b) By the law of sines,

$$a' = 2R'\sin A' = 2\sqrt{2Rr}\cos\frac{A}{2} = 2\sqrt{2\cdot\frac{abc}{4\Delta}\cdot\frac{\Delta}{s}}\cdot\sqrt{\frac{s(s-a)}{bc}} = \sqrt{a(b+c-a)};$$

similarly for b' and c'.

(c) Triangle $A'B'C'$ has area

$$\begin{aligned}\Delta' &= \frac{1}{2}b'c'\sin A' = \frac{1}{2}b'c'\cos\frac{A}{2}\\ &= \frac{1}{2}\sqrt{b(c+a-b)}\cdot\sqrt{c(a+b-c)}\cdot\sqrt{\frac{s(s-a)}{bc}}\\ &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \Delta.\end{aligned}$$

□

Proposition 2. (a) $a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2 - (b-c)^2 - (c-a)^2 - (a-b)^2$.
(b) $a'^2 + b'^2 + c'^2 \le a^2 + b^2 + c^2$.
(c) $a' + b' + c' \le a + b + c$.
(d) $\sin A' + \sin B' + \sin C' \ge \sin A + \sin B + \sin C$.
(e) $R' \le R$.
(f) $r' \ge r$.

In each case, equality holds if and only if ABC is equilateral.

Proof. (a) follows from Lemma 1(b); (b) follows from (a). For (c),

$$\begin{aligned}a' + b' + c' &= \sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)}\\ &\le \frac{b+c}{2} + \frac{c+a}{2} + \frac{a+b}{2}\\ &= a+b+c.\end{aligned}$$

For (d), we have

$$\begin{aligned}&\sin A + \sin B + \sin C\\ =&\frac{1}{2}(\sin B + \sin C + \sin C + \sin A + \sin A + \sin B)\\ =&\sin\frac{B+C}{2}\cos\frac{B-C}{2} + \sin\frac{C+A}{2}\cos\frac{C-A}{2} + \sin\frac{A+B}{2}\cos\frac{A-B}{2}\\ \le&\sin\frac{B+C}{2} + \sin\frac{C+A}{2} + \sin\frac{A+B}{2}\\ =&\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\\ =&\sin A' + \sin B' + \sin C'.\end{aligned}$$

(e) $R' = \frac{a'+b'+c'}{2(\sin A' + \sin B' + \sin C')} \le \frac{a+b+c}{2(\sin A + \sin B + \sin C)} = R$.

(f) $r' = \frac{\Delta'}{s'} \ge \frac{\Delta}{s} = r$. □

Remark. The inequality $R' \leq R$ certainly follows from Euler's inequality $R \geq 2r$. From the direct proof of (e), Euler's inequality also follows (see Theorem 6(b) below).

2. A sequence of triangles

Beginning with a triangle ABC, we repeatedly apply the construction in §1 to obtain a sequence of triangles $(A_nB_nC_n)_{n\in\mathbb{N}}$ with $A_0B_0C_0 \equiv ABC$, and angle measures and sidelengths defined recursively by

$$\begin{aligned} &A_{n+1} = \frac{\pi - A_n}{2}, \quad B_{n+1} = \frac{\pi - B_n}{2}, \quad C_{n+1} = \frac{\pi - C_n}{2}; \\ &a_{n+1} = \sqrt{a_n(b_n + c_n - a_n)}, \quad b_{n+1} = \sqrt{b_n(c_n + a_n - a_n)}, \\ &c_{n+1} = \sqrt{c_n(a_n + b_n - c_n)}. \end{aligned}$$

Denote by s_n, R_n, r_n, Δ_n the semiperimeter, circumradius, inradius, and area of triangle $A_nB_nC_n$. Note that $\Delta_n = \Delta$ for every n.

Lemma 3. *The sequences* $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$ *are convergent and*

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}.$$

Proof. It is enough to consider the sequence $(A_n)_{n\in\mathbb{N}}$. Rewrite the relation $A_{n+1} = \frac{\pi}{2} - \frac{A_n}{2}$ as

$$A_{n+1} - \frac{\pi}{3} = -\frac{1}{2}\left(A_n - \frac{\pi}{3}\right).$$

It follows that the sequence $\left(A_n - \frac{\pi}{3}\right)_{n\in\mathbb{N}}$ is a geometric sequence with common ratio $-\frac{1}{2}$. It converges to 0, giving $\lim_{n\to\infty} A_n = \frac{\pi}{3}$. □

Proposition 4. *The sequence* $(R_n)_{n\in\mathbb{N}}$ *is convergent and* $\lim_{n\to\infty} R_n = \frac{2}{3}\sqrt{\sqrt{3}\Delta}$.

Proof. Since $R_n = \frac{a_nb_nc_n}{4\Delta_n} = \frac{8R_n^3 \sin A_n \sin B_n \sin C_n}{4\Delta_n}$, we have

$$R_n^2 = \frac{\triangle}{2\sin A_n \sin B_n \sin C_n}.$$

The result follows from Lemma 3. □

Proposition 5. *The sequences* $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ *are convergent and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 2\sqrt{\frac{\triangle}{\sqrt{3}}}.$$

Proof. This follows from $a_n = 2R_n \sin A_n$, Lemma 3 and Proposition 4. □

From these basic results we obtain a number of interesting convergent sequences. In each case, the increasing or decreasing property is clear from Proposition 2.

	Sequence		Limit	Reference
(a)	Δ_n	constant	Δ	Lem.1(c)
(b)	$\sin A_n + \sin B_n + \sin C_n$	increasing	$\frac{3\sqrt{3}}{2}$	Prop.2(d), Lem.3
(c)	R_n	decreasing	$\frac{2}{3}\sqrt{\sqrt{3}\Delta}$	Prop.2(e), 4
(d)	s_n	decreasing	$\sqrt{3\sqrt{3}\Delta}$	Prop.2(c), 4
(e)	r_n	increasing	$\frac{1}{3}\sqrt{\sqrt{3}\Delta}$	Prop.2(f)
(f)	$\frac{R_n}{r_n}$	decreasing	2	
(g)	$a_n^2 + b_n^2 + c_n^2$	decreasing	$4\sqrt{3}\Delta$	Prop.2(b), 5
(h)	$a_n^2 + b_n^2 + c_n^2 - (b_n - c_n)^2$ $-(c_n - a_n)^2 - (a_n - b_n)^2$	decreasing	$4\sqrt{3}\Delta$	Prop.2(a, b), 5

3. Geometric inequalities

The increasing or decreasing properties of these sequences, along with their limits, lead easily to a number of famous geometric inequalities [1, 3].

Theorem 6. *The following inequalities hold for an arbitrary angle ABC.*

(a) $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$.

(b) [Euler's inequality] $R \geq 2r$.

(c) [Weitzenböck inequality] $a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta$.

(d) [Hadwiger-Finsler inequality] $a^2+b^2+c^2-(b-c)^2-(c-a)^2-(a-b)^2 \geq 4\sqrt{3}\Delta$.

In each case, equality holds if and only if the triangle is equilateral.

Remark. Weitzenböck's inequality is usually proved as a consequence of the Hadwiger - Finsler's inequality ([2, 4]). Our proof shows that they are logically equivalent.

References

[1] O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasic, *Geometric Inequalities*, Wolters-Noordhoe Publishing, Groningen, 1969.

[2] P. von Finsler and H. Hadwiger, Einige Relationen im Dreieck, *Comment. Math. Helvetici*, 10 (1937) 316–326.

[3] D. S. Mitrinović, J. E. Peĉarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publisher, 1989

[4] R. Weitzenböck, Uber eine Ungleichung in der Dreiecksgeometrie, *Math. Zeit.*, 5 (1919) 137–146.

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