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Some Loci in the Animation of a Sangaku Diagram

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Abstract. In a symmetric partition of a regular n-gon into n congruent subtriangles and a regular n-gon in the center, we determine the loci of the incenter and points of tangency of the incircle a subtriangle.

A famous Sangaku problem ([1, Problem 2.1.7], [2]) asks to partition an equilateral triangle into four subtriangles with congruent incircle (see Figure 1). Ito and Wimmer [3] considered the same problem for general regular polygons (see Figure 2 for the case of a regular pentagon). In this note, we consider a dynamic situation by letting the congruent subtriangles by the sides of the regular polygon vary, and examine the loci of various points in the configuration.

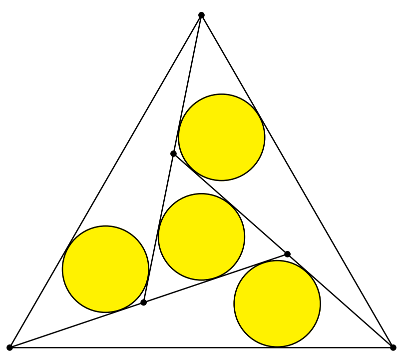

Figure 1

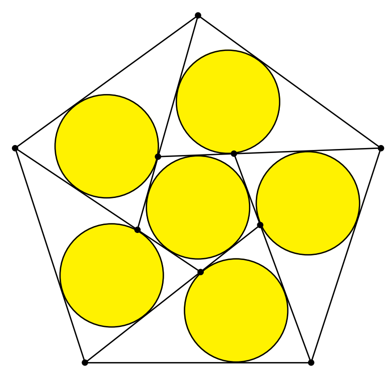

Figure 2

Given a regular n-gon $\mathcal{P} := A_1A_2\cdots A_n$, $k = 1, 2, \ldots, n$, with center O, pass a line $\ell_k(\theta)$ through the vertex A_k such that the directed angle $(A_kA_{k+1}, \ell_k(\theta)) = \theta \in \left(0, \pi - \frac{2\pi}{n}\right)$. Here indices are taken modulo n so that $A_{n+1} = A_1$ etc. Let $A'_k(\theta)$ be the intersection of $\ell_k(\theta)$ and $\ell_{k+1}(\theta)$ (see Figure 3). When there is no danger of confusion, we shall simply write A'_k for $A'_k(\theta)$. Then the regular n-gon $\mathcal{P}$ is partitioned into
(i) n congruent triangles $A'_kA_kA_{k+1}$ for $k = 1, 2, \ldots, n$, and
(ii) a regular n-gon $\mathcal{P}' = A'_1A'_2\cdots A'_n$ at the center.

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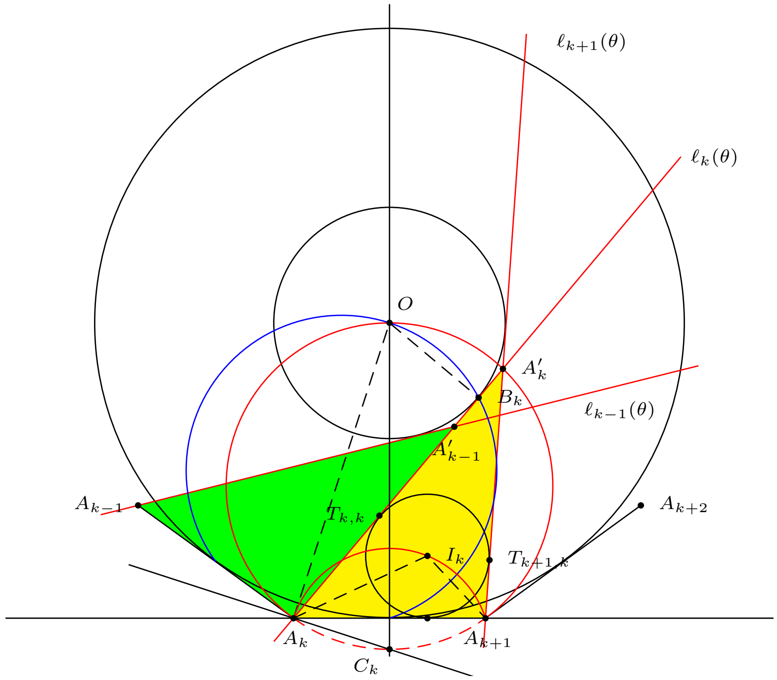

Figure 3

It is clear that the center of the regular n-gon $A'_1A'_2\cdots A'_n$ is the fixed point O. On the other hand, the locus of $A'_k(\theta)$ is the part of the circle OA_kA_{k+1} in the interior of $\mathcal{P}$. This is because

$$\angle A_kA'_kA_{k+1} = \left(\frac{2\pi}{n}+\theta\right) - \theta = \frac{2\pi}{n} = \angle A_kOA_{k+1},$$

from which A_k, A'_k, O, and A_{k+1} are concyclic.

Suppose the incircle of $\mathcal{P}'$ touches the sides $A'_{k-1}A'_k$ at B_k, the midpoint of A'_{k-1} and A'_k on ℓ_k. Since OB_k is perpendicular to $A'_{k-1}A'_k$, it is perpendicular the line $\ell_k(\theta)$ through A_k. Therefore, OB_kA_k is a right angle, and B_k lies on the part of the circle with diameter OA_k inside the regular polygon $\mathcal{P}$. If M_{k-1} and M_k are the midpoints of $A_{k-1}A_k$ and A_kA_{k+1}, then this is the arc of the circle $M_{k-1}OM_k$ in the interior of $\mathcal{P}$.

Now we consider the incircle of triangle $\mathbf{T}_k = A'_kA_kA_{k+1}$, with incenter I_k, and tangent to ℓ_k, ℓ_{k+1} at the points $T_{k,k}$, $T_{k+1,k}$ respectively.

Note that

$$\angle A_kI_kA_{k+1} = \frac{\theta}{2} + \frac{2\pi}{n} + \frac{1}{2}\left(\pi - \frac{2\pi}{n} - \theta\right) = \frac{\pi}{2} - \frac{\pi}{n}$$

is independent of θ. This means that the locus of I_k is the part of a circle through A_k and A_{k+1} in the interior of $\mathcal{P}$. Its center C_k is the intersection of the perpendicular bisector of A_kA_{k+1} and the *external* bisector of angle $A_{k-1}A_kA_{k+1}$.

We summarize these simple results in the following proposition.

Proposition 1. *Let $\mathcal{P} := A_1A_2\cdots A_n$ be a regular n-gon with center O. For $k = 1, 2, \ldots, n$, let $\ell_k(\theta)$ be the line through the vertex A_k such that the directed angle $(A_kA_{k+1}, \ell_k(\theta)) = \theta$ (with indices taken modulo n). As θ varies in $\left(0, \pi - \frac{2\pi}{n}\right)$, the loci of*
(a) *the intersection $A'_k(\theta)$ of $\ell_k(\theta)$ and $\ell_{k+1}(\theta)$ is the part of the circle OA_kA_{k+1} in the interior of $\mathcal{P}$,*
(b) *the point of tangency B_k of the incircle of the regular n-gon $A'_1A'_2\cdots A'_n$ with the line $\ell_k(\theta)$ is the part of the circle $OM_{k-1}M_k$ in the interior of $\mathcal{P}$, M_k being the midpoint A_kA_{k+1},*
(c) *the incenter I_k of triangle $A'_kA_kA_{k+1}$ is the arc of the circle, center C_k, passing through A_{k-1}, C_k being the intersection of the perpendicular bisector of A_kA_{k+1} and the external bisector of angle $A_{k-1}A_kA_{k+1}$.*

We compute some of the lengths in this configuration. In Figure 3, let a be the length of a side of the regular n-gon $\mathcal{P}$. Suppose in triangle $A'_kA_kA_{k+1}$, $A'_kA_{k+1} = b$ and $A_kA'_k = c$. By the law of sines,

$$b = \frac{a}{\sin\frac{2\pi}{n}}\sin\theta,$$
$$c = \frac{a}{\sin\frac{2\pi}{n}}\sin\left(\frac{2\pi}{n}+\theta\right).$$

Theorem 2. *For $k = 1, 2, \ldots, n$, let $T_{k,k}$ and $T_{k+1,k}$ be the points of tangency of the incircle of triangle $A'_kA_kA_{k+1}$ with the lines $\ell_k(\theta)$ and $\ell_{k+1}(\theta)$ respectively. θ varies in $\left(0, \pi - \frac{2\pi}{n}\right)$, the loci of $T_{k,k}$ and $T_{k+1,k}$ are the parts of limaçon inside the regular n-gon $\mathcal{P}$, symmetric with respect to the perpendicular bisector of A_kA_{k+1}.*

Proof. The point of tangency $T_{k,k}$ is the point on $\ell_k(\theta)$ uniquely determined by the length of $A_kT_{k,k}$. Now, in triangle $A'_kA_kA_{k+1}$,

$$\begin{aligned}
A_kT_{k,k} &= \frac{1}{2}(a+c-b)\\
&= \frac{a}{2} + \frac{a}{2\sin\frac{2\pi}{n}}\left(\sin\left(\frac{2\pi}{n}+\theta\right) - \sin\theta\right)\\
&= \frac{a}{2} + \frac{a}{2\sin\frac{2\pi}{n}}\cdot 2\cos\left(\frac{\pi}{n}+\theta\right)\sin\frac{\pi}{n}\\
&= \frac{a}{2} + \frac{a}{2\cos\frac{\pi}{n}}\cdot\cos\left(\frac{\pi}{n}+\theta\right).
\end{aligned}$$

Let C_k be the intersection of the perpendicular bisector of A_kA_{k+1} and the external angle of $A_{k-1}A_kA_{k+1}$. Note that $A_kC_k = \frac{a}{2\cos\frac{\pi}{n}}$. In a polar coordinate system with pole at A_k and polar axis the half line A_kA_{k+1}, the equation

$$\rho = \frac{a}{2\cos\frac{\pi}{n}}\cos\left(\frac{\pi}{n}+\theta\right)$$

represents the circle with A_kC_k as diameter. Therefore,

$$\rho = \frac{a}{2} + \frac{a}{2\cos\frac{\pi}{n}}\cos\left(\frac{\pi}{n}+\theta\right)$$

is a limaçon: If the circle with diameter A_kC_k intersects $\ell_k(\theta)$ at D_k, then $T_{k,k}$ is the point obtained by translating D_k by $\frac{a}{2}$ (along A_kD_k); see Figure 4 for the case of a regular pentagon $A_1A_2A_3A_4A_5$ with $k = 1$.

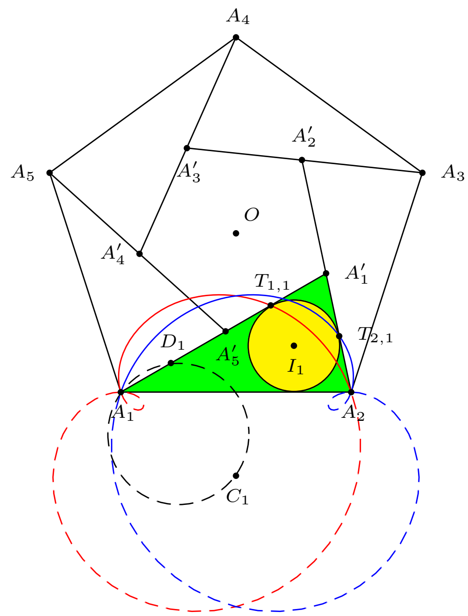

Figure 4

The locus of $T_{k+1,k}$ is clearly the reflection of the locus of $T_{k,k}$ in the perpendicular bisector of A_kA_{k+1}. It is determined by

$$A_{k+1}T_{k+1,k} = \frac{1}{2}(a - c + b) = \frac{a}{2} - \frac{a}{2\cos\frac{\pi}{n}} \cdot \cos\left(\frac{\pi}{n} + \theta\right).$$

This is the limaçon with respect to the circle diameter $A_{k+1}C_k$ and length $\frac{a}{2}$. $\square$

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[3] N. Ito and H. Wimmer, H, A Sangaku-type problem with regular polygons, triangles, and congruent incircles, *Forum Geom.*, 13 (2013) 185–190.

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