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The Golden Ratio and Regular Polygons

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Abstract. We give a simple construction of a chord of the circumcircle of an isosceles triangle whose intersections with the legs of the triangles divide symmetrically the chord in the golden ratio.

Proposition 1. *Let ABC be an isosceles triangle with $AB = AC$, and D, E, F points on BC, CA, AB respectively, such that $BDEF$ is a parallelogram with $BD = DF = FE$. If EF is extended to intersect the circumcircle of ABC at G and H (so that F is between E and G), then F divides EG in the golden ratio.*

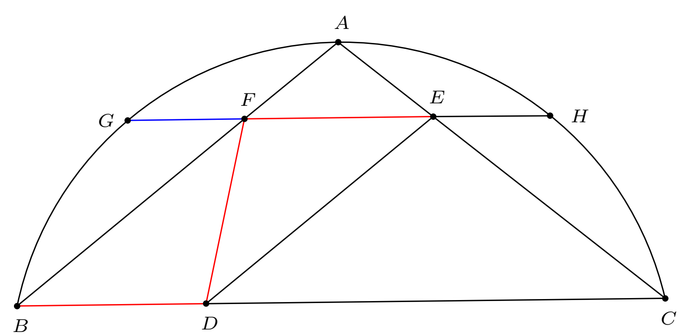

Figure 1

Proof. As EF is parallel to BD, the corresponding angles DBF and EFA are equal. Therefore the isosceles triangles DFB and AFE must be similar. Let $x = EF(= DF = BD)$, $y = FG(= EH)$, $a = AF$, and $b = FB$. The similarity of the triangles DFB and AFE implies $BD : BF = FA : FE$. Thus, $x : b = a : x$, and

$$x^2 = ab. \tag{1}$$

Using the intersecting chords theorem we get $GF \cdot FH = AF \cdot FB$, or

$$y(x + y) = ab. \tag{2}$$

From (1) and (2), $x^2 = (x + y)y$, or

$$\frac{x}{y} = \frac{x + y}{x}.$$

This means that F divides EG in the golden ratio, □

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The construction of the parallelogram can be effected by a central dilation. If the perpendicular bisector of AB intersects BC at D', and the parallelogram $ABD'E'$ is completed, then $BD' = D'A = AE'$. Let BE' intersect AC at E, and the parallel through E to AB and BC intersect BC at D and AB at F respectively, then $BDEF$ is a parallelogram satisfying $BD = DF = FE$ (see Figure 2).

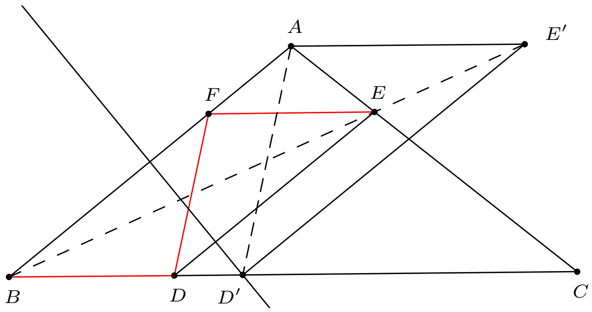

Figure 2

Previous golden ratio constructions based on an equilateral triangle (Odom [1]) or on a square (Tran [2]) are special cases of the present method. Our construction actually gives a simple alternative to the construction in [3], which applies to general regular polygons. In Figure 3, ABC is an isosceles triangle with $AB = AC$, and D, D' are points on BC, E on AB, E' on AC such that $BDE'E$ and $CD'EE'$ are parallelograms with $BD = DE = EE'$ and $CD' = D'E' = E'E$. If B' and C' are the reflections of D in AB and D' in AC respectively, then BB' and CC' are tangents to the circumcircle of ABC at B and C, and E, E' are the trisection points of the segment $B'C'$ (see [3, Proposition 1 and Figure 4]).

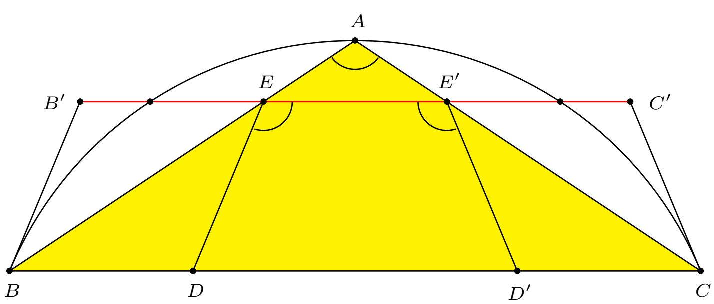

Figure 3

In Figure 3, suppose that AB and AC are consecutive sides of a regular n-gon. Since $\angle DEE' = \angle EE'D' = \angle BAC$, D, E, E', D' are consecutive vertices of a similar regular n-gon (see Figures 4A-C and 5A).

A most interesting case occurs for the pentagons. Figure 5A shows that the smaller regular pentagon has one vertex at the center of the circle circumscribing

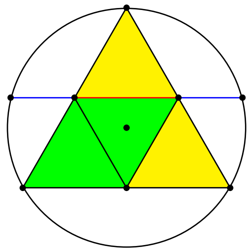

Figure 4A

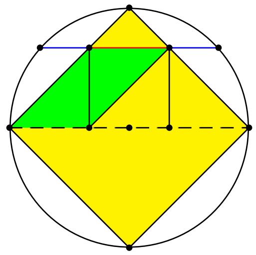

Figure 4B

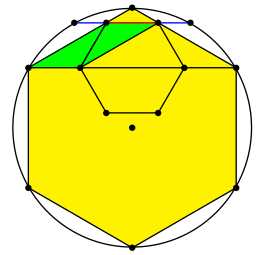

Figure 4C

the given regular pentagon. The areas of the two regular pentagons are in the ratio $1 : 5$. Figure 5B shows a regular pentagon and its reflection about a diameter of its circumcircle parallel to a side. On each side there are two intersection points, dividing it into a solid segment in middle with two dashed segments. In this configuration, each linear segment consisting of a solid segment and a dashed segment is divided in the golden ratio at the junction point.

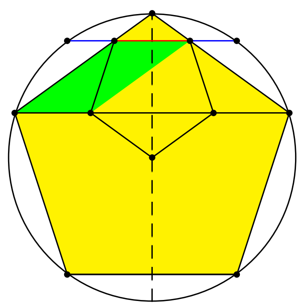

Figure 5A

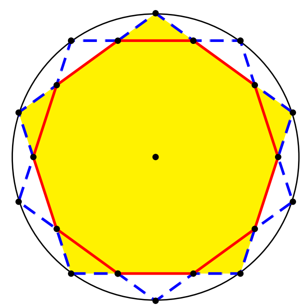

Figure 5B

References

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[3] D. Paunić and P. Yiu, Regular polygons and the golden section, *Forum Geom.*, 16 (2016) 273–281.

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